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SOME HYPERGEOMETRIC RESULTS DERIVED BY THE APPLICATION OF EXPONENTIAL FUNCTIONS

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Abstract— In this paper, we obtain some generating functions for the sum of two ${}_1F_2(.)$ hypergeometric functions, for ${}_2F_2(.)$ hypergeometric function and for hypergeometric polynomials ${}_3F_0(.)$ by using series decomposition technique, series rearrangement method. Furthermore we obtain the hypergeometric forms of sine and cosine functions.

Keywords— *Series rearrangement technique, generating functions, generalized hypergeometric functions.*

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I. 1INTRODUCTION AND PRELIMINARIES

Throughout the present work, we use the following standard notations: $\mathbb{N} := \{1,2,3,\ldots\}$, $\mathbb{N}_0 := \{0,1,2,3,\ldots\} = \mathbb{N} \cup \{0\}$ and $\mathbb{Z}^- := \{-1,-2,-3,\ldots\} = \mathbb{Z}_0^- \setminus \{0\}$. Here, as usual, $\mathbb{Z}$ denotes the set of integers, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{R}_+$, $\mathbb{R}_-$ denote the sets of positive and negative real numbers respectively and $\mathbb{C}$ denotes the set of complex numbers.

The Pochhammer symbol (or the shifted factorial) $(\lambda)_\nu$ $(\lambda, \nu \in \mathbb{C})$ is defined, in terms of the familiar Gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0; \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\ldots(\lambda+n-1) & (\nu = n \in \mathbb{N}; \lambda \in \mathbb{C}) \end{cases} \tag{1.1}$$

it being understood *conventionally* that $(0)_0 = 1$ and assumed tacitly that the Gamma quotient exists.

The following results will be required in our present investigations:

The generalized hypergeometric function of one variable with p numerator parameters and q denominator parameters is defined by

$${}_pF_q\left[\begin{matrix}\alpha_1, \alpha_2, \ldots, \alpha_p \,; \\ \beta_1, \beta_2, \ldots, \beta_q \,;\end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n (\alpha_2)_n \ldots (\alpha_p)_n}{(\beta_1)_n (\beta_2)_n \ldots (\beta_q)_n} \frac{z^n}{n!}. \tag{1.2}$$

Here p and q are positive integers or zero (interpreting an empty product as 1), and we assume that the variable z, the numerator parameters $\alpha_1, \alpha_2, \ldots, \alpha_p$ and the denominator parameters $\beta_1, \beta_2, \ldots, \beta_q$ take on complex values, provided that $\beta_j \neq 0,-1,-2,\ldots;$ $j = 1,2,\ldots,q$.

Supposing that none of the numerator parameters is zero or a negative integer (otherwise the question of convergence will not arise), and with the usual restriction on β_j, the ${}_pF_q$ series in (1.2):

(i) converges for $|z| < \infty$ if $p \leq q$,

(ii) converges for $|z| < 1$, if $p = q+1$,

(iii) diverges for all z, $z \neq 0$, if $p > q+1$.

Furthermore, if we set

$$\omega = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j, \quad (1.3)$$

It is known that the ${}_pF_q$ series, with $p = q+1$, is

(I) absolutely convergent for $|z| = 1$, if $\Re(\omega) > 0$,

(II) conditionally convergent for $|z| = 1$, $|z| \neq 1$, if $-1 < \Re(\omega) \leq 0$,

(III) divergent for $|z| = 1$, if $\Re(\omega) \leq -1$.

The idea of decomposition of a power series into its even and odd terms [4, p.200(1)], exhibited by the elementary identity

$$\sum_{n=0}^{\infty} \Psi(n) = \sum_{n=0}^{\infty} \Psi(2n) + \sum_{n=0}^{\infty} \Psi(2n+1), \quad (1.4)$$

is at least as old as the series themselves.

Closed form of reduction formula [3, p.70 Q.N.10] for Gauss function is given by

$${}_2F_1\left[\begin{matrix} a, & a+\frac{1}{2}; \\ 2a+1 & ; \end{matrix} z\right] = \left(\frac{2}{1+\sqrt{(1-z)}}\right)^{2a}, \quad (1.5)$$

where $2a+1 \neq 0, -1, -2, -3, \ldots$; $|z| < 1$.

Binomial theorem in terms of hypergeometric function is given by

$$(1-z)^{-\alpha} = {}_1F_0\left[\begin{matrix} \alpha; \\ -; \end{matrix} z\right] = \sum_{n=0}^{\infty} (\alpha)_n \frac{z^n}{n!} ; \qquad |z| < 1, \quad \alpha \in \mathbb{C}. \quad (1.6)$$

Double series identity [3, p.57 (7); see also 4, p.100 (3)] is given by

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{\left[\frac{n}{2}\right]} A(k, n-2k) \quad (1.7)$$

where, and in what follows, $[x]$ denotes the greatest integer in x.

Gauss summation theorem [3, p.49 example] is given by

$${}_2F_1\left[\begin{matrix} -\frac{n}{2}, \frac{-n+1}{2}; \\ b+\frac{1}{2} \quad ; \end{matrix} 1\right] = \frac{2^n (b)_n}{(2b)_n} \quad (1.8)$$

where $b+\frac{1}{2} \neq 0, -1, -2, -3, \ldots$ and $n \in \mathbb{N}_0$.

$$\sinh^{-1}(t) = \ell_n(t+\sqrt{(1+t^2)}) \quad (1.9)$$

$$\sin^{-1}(it) = i\sinh^{-1}(t) \quad (1.10)$$

$$\sin^{-1}(it) = i\ell_n(t+\sqrt{(1+t^2)}) \quad (1.11)$$

$$\sin^{-1}(-t) = i\ell_n(it+\sqrt{(1-t^2)}) \quad (1.12)$$

$$x\sin^{-1}(t) = -ix\ell_n(it+\sqrt{(1-t^2)}) \quad (1.13)$$

Motivated by the work on generating functions recorded in beautiful monographs of Erde'lyi etal. [1, Chapter 19], McBride [2, Chapter 1] and Srivastava-Manocha [4, Chapters 2,3], we obtain some generating relations associated with the combination of two ${}_1F_2(.)$ hypergeometric functions, one ${}_2F_2(.)$ hypergeometric function and two hypergeometric polynomials ${}_3F_0(.)$, by the applications of some exponential functions as generating functions. Furthermore we obtain the hypergeometric forms of sine and cosine functions.

II. FIRST GENERATING RELATION

When $|t| < 1$; $\alpha, \alpha+\frac{1}{2} \neq 0, -1, -2, -3, \ldots$; for all finite values of x, then following generating relation holds true

$$(1-t)^{-\alpha}\exp\left[\frac{xt}{(1-t)+\sqrt{(1-t)}}+x\right]$$

$$=\sum_{n=0}^{\infty}\left\{(\alpha)_n\ {}_1F_2\left[\begin{matrix}\alpha+n\ ; \\ \alpha,\ \frac{1}{2}\ ;\end{matrix}\ \frac{x^2}{4}\right]+x\left(\alpha+\frac{1}{2}\right)_n\ {}_1F_2\left[\begin{matrix}\alpha+\frac{1}{2}+n\ ; \\ \alpha+\frac{1}{2},\ \frac{3}{2}\ ;\end{matrix}\ \frac{x^2}{4}\right]\right\}\frac{t^n}{n!} \tag{2.1}$$

Proof: Consider the expression

$$(1-t)^{-\alpha}\exp\left[\frac{xt}{(1-t)+\sqrt{(1-t)}}\right]=(1-t)^{-\alpha}\exp\left[\frac{x\{1-(1-t)\}}{\sqrt{(1-t)}(1+\sqrt{(1-t)})}\right]$$

$$=(1-t)^{-\alpha}\exp\left[\frac{x\{1-\sqrt{(1-t)}\}\{1+\sqrt{(1-t)}\}}{\sqrt{(1-t)}\{1+\sqrt{(1-t)}\}}\right]$$

$$=\exp(-x)(1-t)^{-\alpha}\exp\left(\frac{x}{\sqrt{1-t}}\right)$$

$$=\exp(-x)\sum_{r=0}^{\infty}\frac{x^r}{r!}(1-t)^{-\left(\alpha+\frac{r}{2}\right)} \tag{2.2}$$

Now using hypergeometric form (1.6) of Binomial theorem, we get

$$(1-t)^{-\alpha}\exp\left[\frac{xt}{(1-t)+\sqrt{(1-t)}}\right]=\exp(-x)\sum_{r=0}^{\infty}\frac{x^r}{r!}\ {}_1F_0\left[\begin{matrix}\alpha+\frac{r}{2}\ ; \\ -\ ;\end{matrix}\ t\right];\quad |t|<1$$

$$=\exp(-x)\sum_{r=0}^{\infty}\frac{x^r}{r!}\sum_{n=0}^{\infty}\frac{\left(\alpha+\frac{r}{2}\right)_n}{n!}t^n$$

$$=\exp(-x)\sum_{n=0}^{\infty}\left\{\sum_{r=0}^{\infty}\left(\frac{2\alpha+r}{2}\right)_n\frac{x^r}{r!}\right\}\frac{t^n}{n!} \tag{2.3}$$

Now applying the series decomposition identity (1.4) in equation (2.3), we have,

$$(1-t)^{-\alpha}\exp\left[\frac{xt}{(1-t)+\sqrt{(1-t)}}\right]$$

$$=\exp(-x)\sum_{n=0}^{\infty}\left\{\sum_{r=0}^{\infty}(\alpha+r)_n\frac{x^{2r}}{(2r)!}+x\sum_{r=0}^{\infty}\left(\alpha+r+\frac{1}{2}\right)_n\frac{x^{2r}}{(2r+1)!}\right\}\frac{t^n}{n!}$$

$$= \exp(-x)\sum_{n=0}^{\infty}\left\{\sum_{r=0}^{\infty}\frac{\Gamma(\alpha+n+r)}{\Gamma(\alpha+r)}\frac{x^{2r}}{2^{2r}\left(\frac{1}{2}\right)_r r!}+x\sum_{r=0}^{\infty}\frac{\Gamma(\alpha+n+r+\frac{1}{2})}{\Gamma(\alpha+r+\frac{1}{2})}\frac{x^{2r}}{2^{2r}\left(\frac{3}{2}\right)_r r!}\right\}\frac{t^n}{n!}$$

$$= \exp(-x)\sum_{n=0}^{\infty}\left\{\sum_{r=0}^{\infty}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\frac{(\alpha+n)_r}{(\alpha)_r}\frac{x^{2r}}{2^{2r}\left(\frac{1}{2}\right)_r r!}+x\sum_{r=0}^{\infty}\frac{\Gamma(\alpha+n+\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})}\frac{(\alpha+n+\frac{1}{2})_r}{(\alpha+\frac{1}{2})_r}\frac{x^{2r}}{2^{2r}\left(\frac{3}{2}\right)_r r!}\right\}\frac{t^n}{n!}$$

$$= \exp(-x)\sum_{n=0}^{\infty}\left\{(\alpha)_n\ {}_1F_2\left[\begin{matrix}\alpha+n\ ; \\ \alpha,\ \frac{1}{2}\ ;\end{matrix}\ \frac{x^2}{4}\right]+x\left(\alpha+\frac{1}{2}\right)_n\ {}_1F_2\left[\begin{matrix}\alpha+\frac{1}{2}+n\ ; \\ \alpha+\frac{1}{2},\ \frac{3}{2}\ ;\end{matrix}\ \frac{x^2}{4}\right]\right\}\frac{t^n}{n!} \quad (2.4)$$

which completes the proof of generating relation (2.1).

III. SECOND GENERATING RELATION

When $|t|<1$; $\alpha \neq 0,-1,-2,-3,\ldots$; for all finite values of x, then following generating relation holds true

$$(1-t)^{-\alpha}\exp\left[\frac{x^2t(t-2)}{(1-t)^2}-x^2\right]=\sum_{n=0}^{\infty}(\alpha)_n\ {}_2F_2\left[\begin{matrix}\frac{\alpha+n}{2},\frac{\alpha+1+n}{2}\ ; \\ \frac{\alpha}{2},\ \frac{1+\alpha}{2}\ ;\end{matrix}\ -x^2\right]\frac{t^n}{n!} \quad (3.1)$$

Proof: Consider the expression

$$(1-t)^{-\alpha}\exp\left[\frac{x^2t(t-2)}{(1-t)^2}\right]=(1-t)^{-\alpha}\exp\left[\frac{x^2[(1-t)^2-1]}{(1-t)^2}\right]$$

$$=(1-t)^{-\alpha}\exp(x^2)\exp\left[\frac{-x^2}{(1-t)^2}\right]$$

$$=\exp(x^2)\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{r!}(1-t)^{-(2r+\alpha)} \quad (3.2)$$

Now using hypergeometric form (1.6) of Binomial theorem, we get

$$(1-t)^{-\alpha}\exp\left[\frac{x^2t(t-2)}{(1-t)^2}\right]=\exp(x^2)\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{r!}\ {}_1F_0\left[\begin{matrix}2r+\alpha\ ; \\ -\ ;\end{matrix}\ t\right]\ ;\ |t|<1$$

$$=\exp(x^2)\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}\Gamma(2r+\alpha+n)}{r!\Gamma(2r+\alpha)}\frac{t^n}{n!}$$

$$= \exp(x^2)\sum_{n=0}^{\infty}(\alpha)_n \sum_{r=0}^{\infty} \frac{(\frac{\alpha+n}{2})_r (\frac{\alpha+1+n}{2})_r (-x^2)^r}{(\frac{\alpha}{2})_r (\frac{1+\alpha}{2})_r r!} \frac{t^n}{n!}$$

$$= \exp(x^2)\sum_{n=0}^{\infty}(\alpha)_n \; {}_2F_2\left[\begin{matrix} \frac{\alpha+n}{2}, \frac{\alpha+1+n}{2} & ; \\ \frac{\alpha}{2}, \frac{1+\alpha}{2} & ; \end{matrix} -x^2\right] \frac{t^n}{n!} \tag{3.3}$$

which completes the proof of generating relation (3.1).

IV. THIRD GENERATING RELATIONS

$$\sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n+2}}{(2n+2)!} \; {}_3F_0\left[\begin{matrix} -n, -2n-2, -n-\frac{1}{2} & ; \frac{-4}{x^2} \\ - & ; \end{matrix}\right] t^{n+1} = \cosh\left[\frac{x\{1-\sqrt{(1-t)}\}}{\sqrt{t}}\right] \tag{4.1}$$

$$\sum_{n=0}^{\infty} \frac{\left(\frac{x}{2}\right)^{2n+1}}{(2n+1)!} \; {}_3F_0\left[\begin{matrix} -n, -2n-1, -n+\frac{1}{2} & ; \frac{-4}{x^2} \\ - & ; \end{matrix}\right] t^{n} = \frac{1}{\sqrt{t}} \sinh\left[\frac{x\{1-\sqrt{(1-t)}\}}{\sqrt{t}}\right] \tag{4.2}$$

Proof: Consider the expression

$$\exp\left[\frac{x(\sqrt{(1-t^2)}-1)}{t}\right] = \exp\left[x\left\{\frac{(\sqrt{(1-t^2)}-1)(\sqrt{(1-t^2)}+1)}{t(\sqrt{(1-t^2)}+1)}\right\}\right]$$

$$= \exp\left[\frac{-xt}{1+\sqrt{(1-t^2)}}\right]$$

$$= \sum_{n=0}^{\infty} \frac{(-xt)^n}{(1+\sqrt{1-t^2})^n n!} \tag{4.3}$$

Now using the reduction formula (1.5) for $\{1+\sqrt{1-t^2}\}^{-n}$ in equation (4.3), we get

$$\exp\left[\frac{x(\sqrt{(1-t^2)}-1)}{t}\right] = \sum_{n=0}^{\infty} \frac{(-xt)^n}{n!} \frac{1}{2^n} \; {}_2F_1\left[\begin{matrix} \frac{n+1}{2}, \frac{n}{2} & ; t^2 \\ n+1 & ; \end{matrix}\right] ; \; |t^2|<1$$

$$= \sum_{n=0}^{\infty} \frac{(-x)^n}{n! 2^n} \sum_{m=0}^{\infty} \frac{\left(\frac{n+1}{2}\right)_m \left(\frac{n}{2}\right)_m t^{(n+2m)}}{(n+1)_m m!}$$

$$= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(-x)^n (n)_{2m} t^{n+2m}}{m! 2^{n+2m} (1)_{n+m}} \tag{4.4}$$

Now using the double series identity (1.7) in equation (4.4), we get

$$\exp\left[\frac{x(\sqrt{(1-t^2)}-1)}{t}\right]=\sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]}\frac{(-x)^{n-2m}(n-2m)_{2m}t^n}{m!2^n(1)_{n-m}}$$

$$=\sum_{n=0}^{\infty}\frac{\left(\frac{-x}{2}\right)^n}{n!}\left\{\sum_{m=0}^{[\frac{n}{2}]}\frac{(-n)_m\left(\frac{-n+1}{2}\right)_m\left(\frac{-n+2}{2}\right)_m\left(\frac{-4}{x^2}\right)^m}{m!}\right\}t^n \tag{4.5}$$

Now applying the series decomposition identity (1.4) in equation (4.5), we have

$$\exp\left[\frac{x\{\sqrt{(1-t^2)}-1\}}{t}\right]=\sum_{n=0}^{\infty}\frac{\left(-\frac{x}{2}\right)^{2n}}{(2n)!}\left\{\sum_{m=0}^{n}\frac{(-2n)_m\left(-n+\frac{1}{2}\right)_m\left(-(n-1)\right)_m}{m!}\left(-\frac{4}{x^2}\right)^m\right\}t^{2n}+$$

$$+\sum_{n=0}^{\infty}\frac{\left(-\frac{x}{2}\right)^{2n+1}}{(2n+1)!}\left\{\sum_{m=0}^{n}\frac{(-2n-1)_m(-n)_m\left(-n+\frac{1}{2}\right)_m}{m!}\left(-\frac{4}{x^2}\right)^m\right\}t^{2n+1} \tag{4.6}$$

Since $(-n+1)_m \neq 0$ when $m=0,1,2,3,\ldots,n-1$; $(-n+1)_n=0$, further empty sum is treated as zero, therefore we have

$$\exp\left[\frac{x\{\sqrt{(1-t^2)}-1\}}{t}\right]=\sum_{n=1}^{\infty}\frac{\left(\frac{x}{2}\right)^{2n}}{(2n)!}\left\{\sum_{m=0}^{n-1}\frac{(-2n)_m\left(-n+\frac{1}{2}\right)_m\left(-n+1\right)_m}{m!}\left(-\frac{4}{x^2}\right)^m\right\}t^{2n}-$$

$$-\sum_{n=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2n+1}}{(2n+1)!}\left\{\sum_{m=0}^{n}\frac{(-2n-1)_m(-n)_m\left(-n+\frac{1}{2}\right)_m}{m!}\left(-\frac{4}{x^2}\right)^m\right\}t^{2n+1} \tag{4.7}$$

$$\exp\left[\frac{x\{\sqrt{(1-t^2)}-1\}}{t}\right]=t^2\sum_{n=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2n+2}}{(2n+2)!}\left\{\sum_{m=0}^{n}\frac{(-2n-2)_m\left(-n-\frac{1}{2}\right)_m(-n)_m}{m!}\left(-\frac{4}{x^2}\right)^m\right\}t^{2n}-$$

$$-t\sum_{n=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2n+1}}{(2n+1)!}\left\{\sum_{m=0}^{n}\frac{(-2n-1)_m(-n)_m\left(-n+\frac{1}{2}\right)_m}{m!}\left(-\frac{4}{x^2}\right)^m\right\}t^{2n} \quad (4.8)$$

$$\exp\left[\frac{x\{\sqrt{(1-t^2)}-1\}}{t}\right]=t^2\sum_{n=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2n+2}}{(2n+2)!}\,{}_3F_0\left[\begin{matrix}-n,-2n-2,-n-\frac{1}{2}\,; & \frac{-4}{x^2}\\ -\quad ; & \end{matrix}\right](t^2)^n-$$

$$-t\sum_{n=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2n+1}}{(2n+1)!}\,{}_3F_0\left[\begin{matrix}-n,-2n-1,-n+\frac{1}{2}\,; & \frac{-4}{x^2}\\ -\quad ; & \end{matrix}\right](t^2)^n \quad (4.9)$$

Suppose x is real, put $t=iT$ or $t^2=-T^2$ in equation (4.9) and equating real and imaginary parts, we have

$$-\frac{1}{T^2}\cos\left[\frac{x\{1-\sqrt{(1+T^2)}\}}{T}\right]=\sum_{n=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2n+2}}{(2n+2)!}\,{}_3F_0\left[\begin{matrix}-n,-2n-2,-n-\frac{1}{2}\,; & \frac{-4}{x^2}\\ -\quad ; & \end{matrix}\right](-T^2)^n \quad (4.10)$$

$$-\frac{1}{T}\sin\left[\frac{x\{1-\sqrt{(1+T^2)}\}}{T}\right]=\sum_{n=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2n+1}}{(2n+1)!}\,{}_3F_0\left[\begin{matrix}-n,-2n-1,-n+\frac{1}{2}\,; & \frac{-4}{x^2}\\ -\quad ; & \end{matrix}\right](-T^2)^n \quad (4.11)$$

Put $T=i\sqrt{t}$ or $T^2=-t$ in equations (4.10) and (4.11), we get

$$\frac{1}{t}\cos\left[\frac{ix\{-1+\sqrt{(1-t)}\}}{\sqrt{t}}\right]=\sum_{n=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2n+2}}{(2n+2)!}\,{}_3F_0\left[\begin{matrix}-n,-2n-2,-n-\frac{1}{2}\,; & \frac{-4}{x^2}\\ -\quad ; & \end{matrix}\right]t^n \quad (4.12)$$

$$-\frac{1}{i\sqrt{t}}\sin\left[\frac{ix\{-1+\sqrt{(1-t)}\}}{\sqrt{t}}\right]=\sum_{n=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2n+1}}{(2n+1)!}\,{}_3F_0\left[\begin{matrix}-n,-2n-1,-n+\frac{1}{2}\,; & \frac{-4}{x^2}\\ -\quad ; & \end{matrix}\right]t^n \quad (4.13)$$

After simplification we get the generating relations (4.1) and (4.2).

V. SOME HYPERGEOMETRIC FORMS

$$\exp[x\sin^{-1}(t)]={}_2F_1\left[\begin{matrix}\frac{-ix}{2},\frac{ix}{2}\,; & t^2\\ \frac{1}{2}\,; & \end{matrix}\right]+xt\,{}_2F_1\left[\begin{matrix}\frac{1+ix}{2},\frac{-ix+1}{2}\,; & t^2\\ \frac{3}{2}\,; & \end{matrix}\right] \quad (5.1)$$

$$\cos(x\theta) = {}_2F_1\left[\begin{matrix} \frac{-x}{2}, \frac{x}{2} & ; \\ \frac{1}{2} & ; \end{matrix} \sin^2\theta\right] \tag{5.2}$$

$$\sin(x\theta) = x\sin\theta \; {}_2F_1\left[\begin{matrix} \frac{1-x}{2}, \frac{x+1}{2} & ; \\ \frac{3}{2} & ; \end{matrix} \sin^2\theta\right] \tag{5.3}$$

Proof: Taking exponential of both sides of the equation (1.13), we get

$$\exp[x\sin^{-1}(t)] = \exp[-ix\ell_n(it+\sqrt{(1-t^2)})] = (\sqrt{(1-t^2)})^{-ix}\left[1+\frac{it}{\sqrt{(1-t^2)}}\right]^{-ix} \tag{5.4}$$

Now using hypergeometric form (1.6) of Binomial theorem, we get

$$\exp[x\sin^{-1}(t)] = (\sqrt{(1-t^2)})^{-ix}{}_1F_0\left[\begin{matrix} ix & ; \\ - & ; \end{matrix} \frac{-it}{\sqrt{(1-t^2)}}\right] \; ; \; \left|\frac{-it}{\sqrt{(1-t^2)}}\right| < 1$$

$$= \sum_{n=0}^{\infty} \frac{(ix)_n(-i)^n t^n (\sqrt{1-t^2})^{-ix}}{(\sqrt{1-t^2})^n n!}$$

$$= \sum_{n=0}^{\infty} \frac{(ix)_n(-i)^n t^n}{n!} \; {}_1F_0\left[\begin{matrix} \frac{n+ix}{2} & ; \\ - & ; \end{matrix} t^2\right] \; ; \; |t^2| < 1$$

$$= \sum_{n=0}^{\infty} \frac{(ix)_n(-i)^n}{n!} \sum_{m=0}^{\infty} \frac{(\frac{n+ix}{2})_m t^{n+2m}}{m!} \tag{5.5}$$

Now using double series identity (1.7) in equation (5.5), we get

$$\exp[x\sin^{-1}(t)] = \sum_{n=0}^{\infty}\sum_{m=0}^{[\frac{n}{2}]}\left\{\frac{(ix)_{n-2m}(-i)^{n-2m}(\frac{n-2m+ix}{2})_m}{(n-2m)!m!}\right\}t^n$$

$$= \sum_{n=0}^{\infty} \frac{(ix)_n(-i)^n t^n}{n!} \; {}_2F_1\left[\begin{matrix} \frac{-n}{2}, \frac{-n+1}{2} & ; \\ \frac{-ix-n}{2}+\frac{1}{2} & ; \end{matrix} 1\right] \tag{5.6}$$

Now applying Gauss summation theorem (1.8) in equation (5.6), we obtain

$$\exp[x\sin^{-1}(t)] = \sum_{n=0}^{\infty} \frac{(ix)_n(-i)^n t^n}{n!} \frac{2^n\left(\frac{-ix-n}{2}\right)_n}{(-ix-n)_n} = \sum_{n=0}^{\infty}\left\{\frac{(ix)_n(2i)^n\Gamma(\frac{-ix+n}{2})}{(1+ix)_n n!\Gamma(\frac{-ix-n}{2})}\right\}t^n \tag{5.7}$$

Now using the series decomposition identity (1.4) in equation (5.7), we have

$$\exp[x\sin^{-1}(t)] = \sum_{n=0}^{\infty}\left\{\frac{(ix)_{2n}(2i)^{2n}\Gamma(\frac{-ix+2n}{2})}{(1+ix)_{2n}(2n)!\Gamma(\frac{-ix-2n}{2})}\right\}t^{2n} + \sum_{n=0}^{\infty}\left\{\frac{(ix)_{2n+1}(2i)^{2n+1}\Gamma(\frac{-ix+2n+1}{2})}{(1+ix)_{2n+1}(2n+1)!\Gamma(\frac{-ix-2n-1}{2})}\right\}t^{2n+1}$$

$$= \sum_{n=0}^{\infty}\left\{\frac{(ix)_{2n}(2i)^{2n}\Gamma(\frac{-ix+2n}{2})}{(1+ix)_{2n}(2n)!\Gamma(\frac{-ix-2n}{2})}\right\}(t^2)^n + \frac{(ix)(2i)t\Gamma(\frac{-ix+1}{2})}{(1+ix)\Gamma(\frac{-ix-1}{2})}\sum_{n=0}^{\infty}\left\{\frac{(1+ix)_{2n}(2i)_{2n}(\frac{-ix+1}{2})_n}{(2+ix)_{2n}(2n+1)!(\frac{-ix-1}{2})_{-n}}\right\}(t^2)^n \tag{5.8}$$

After simplification we get the hypergeometric form (5.1).

Replacing x by ix in equation (5.1) and equating real and imaginary parts, we have

$$\cos(x\sin^{-1}(t)) = {}_2F_1\left[\begin{matrix}\frac{-x}{2},\frac{x}{2} & ; \\ \frac{1}{2} & ;\end{matrix}\; t^2\right] \tag{5.9}$$

$$\sin(x\sin^{-1}(t)) = xt\; {}_2F_1\left[\begin{matrix}\frac{1-x}{2},\frac{x+1}{2} & ; \\ \frac{3}{2} & ;\end{matrix}\; t^2\right] \tag{5.10}$$

Put $\sin^{-1}(t) = \theta$ or $t = \sin\theta$ in equations (5.9) and (5.10), we get the results (5.2) and (5.3) respectively.

We conclude our present investigation, by observing that several other generating relations and hypergeometric forms can be obtained by using series rearrangement technique and hypergeometric approach.

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