SANGAM International Journal of Multidisciplinary Research

Vol. 02, Issue 01, September (2024) E-ISSN: 2584-086X

THE INTEGRAL TRANSFORM OF PRODUCT OF SPECIAL FUNCTIONS

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Abstract— This paper refers to the study of product of Special G function and Generalized Mittag – Leffler function, we derive various integral transform, including Euler transform, Laplace transform, Whittaker transform, Hankel transform. Some results are expressed in terms of generalized Wright function. The transforms found here are likely to find useful in problem of Sciences, engineering and technology.

Keywords*— Euler transform, Laplace transform, Whittaker transform, Hankel transform, Fractional Fourier transform, Generalized Wright function, Special G function, Generalized Mittag- Leffler function.*

2020 Mathematics Subject Classifications: *26A33, 44A20, 33C20, 33E50*

II. INTRODUCTION

Throughout this paper, R and C denote the sets of real and complex numbers, respectively. Also $R^{+} = (0,\infty)$, $N_0 = \{0, 1, \ldots\}$ and $Z^{-} = \{-1, -2, \ldots\}$.

Definition 1.1 Special G function:

The special $G_{\rho,\eta,\gamma}[a,z]$ is defined by [1, 2] as

$$G_{\rho,\eta,\gamma}[a,z]=z^{\gamma\rho-\eta-1}\sum_{n=0}^{\infty}\frac{(\gamma)_n\left(az^{\rho}\right)^n}{\Gamma(n\rho+\rho\gamma-\eta)n!} \tag{1}$$

Definition 1.2 Generalized Mittag- Leffler Function

Gosta Mittag – Leffler the Swedish mathematician introduced the term Gosta Mittag – Leffler function i.e. , Mittag – Leffler function is defined [3] as

$$E_{\gamma}(d)=\sum_{n=0}^{\infty}\frac{(d)^n}{\Gamma(\gamma n+1)} \qquad \left(d\in C;\quad R(\gamma)>0\right)$$

where is a gamma function , after this Wiman generalized the Mittag – Leffler function as follows,

$$E_{\gamma,\nu}(d)=\sum_{n=0}^{\infty}\frac{(d)^n}{\Gamma(\gamma n+\nu)} \qquad \left(d\in C;\quad \min(R(\gamma)R(\nu))>0\right)$$

there are number of ways in which Mittag- Leffler function

$$E_{\lambda,\beta}^{\phi}[z]=\sum_{m=0}^{\infty}\frac{(\phi)_m}{\Gamma(\lambda m+\beta)}\frac{z^m}{m!} \tag{2}$$

where $\lambda, \beta, \phi \in C \quad (R(\lambda) > 0)$

Definition 1.3 Product of G function and Mittag - Leffler function

$$G_{\rho,\eta,\gamma}[a,z] \times E_{\lambda,\beta}^{\phi}[z] = z^{\gamma\rho-\eta-1} \sum_{n=0}^{\infty} \frac{(\gamma)_n (az^{\rho})^n}{\Gamma(n\rho + \rho\gamma - \eta) n!} \sum_{m=0}^{\infty} \frac{(\phi)_m}{\Gamma(\lambda m + \beta)} \frac{z^m}{m!}$$

let m = n = k then

$$G_{\rho,\eta,\gamma}[a,z] \times E_{\lambda,\beta}^{\phi}[z] = z^{\gamma\rho-\eta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k (az^{\rho})^k}{\Gamma(k\rho + \rho\gamma - \eta) k!} \frac{(\phi)_k}{\Gamma(\lambda k + \beta)} \frac{z^k}{k!} \tag{3}$$

Definition 1.4 Fox – Wright Generalized Hypergeometric Function

In 1933, E.M. Wright defined a more interesting generalized hypergeometric function of one variable [4] and further generalizations of the series ${}_pF_q$ were given by Fox [5] and Wright [6,7,8] ;

$$ {}_p\psi_q(z) = {}_p\psi_q\left[\left.\begin{matrix}(\alpha_1, A_1),\ldots\ldots(\alpha_p, A_p)\\(\beta_1, B_1),\ldots\ldots(\beta_q, B_q)\end{matrix}\right| z\right]$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1 n)\Gamma(\alpha_2 + A_2 n)\ldots\ldots\Gamma(\alpha_p + A_p n)}{\Gamma(\beta_1 + B_1 n)\Gamma(\beta_2 + B_2 n)\ldots\ldots\Gamma(\beta_q + B_q n)} \frac{z^n}{n!} \tag{4}$$

where the coefficients $A_1, \ldots\ldots, A_p \in R^+$ and $B_1, \ldots\ldots, B_q \in R^+$ such that

$$1 + \sum_{j=1}^{q} B_j - \sum_{i=1}^{p} A_i \geq 0$$

for suitably bounded values of $|z|$. $\alpha_1, \alpha_2, \ldots, \alpha_p, \beta_1, \beta_2, \ldots, \beta_q$ are complex parameters.

The Fox - Wright function is a special case of the Fox – H function as [9]

$$ {}_p\psi_q\left[\left.\begin{matrix}(\alpha_1, A_1),\ldots\ldots(\alpha_p, A_p)\\(\beta_1, B_1),\ldots\ldots(\beta_q, B_q)\end{matrix}\right| z\right] = H_{p,q+1}^{1,p}\left[-z\left|\begin{matrix}(1-\alpha_1, A_1),\ldots\ldots(1-\alpha_p, A_p)\\(1-\beta_1, B_1),\ldots\ldots(1-\beta_q, B_q)\end{matrix}\right.\right] \tag{5}$$

Definition 1.5 The Euler Transform ([10], see also [11])

The Euler transform of a function f(z) is defined as

$$B\{f(z); a, b\} = \int_0^1 z^{a-1}(1-z)^{b-1} f(z)\, dz$$
$$a, b \in C, R(a) > 0, R(b) > 0 \tag{6}$$

Definition 1.6 The Laplace Transform ([10])

The Laplace transform of a function $f(t)$, is given by the equation

$$F(s) = L\{f(t); s\} = \int_0^{\infty} e^{-st} f(t)\, dt$$
$$R(s) > 0 \tag{7}$$

Provided that the integral is convergent for $t > 0$ and of exponential order as $t \to \infty$.

Also $\int_0^\infty e^{-st} t^{p-1} dt = \frac{\Gamma(p)}{s^p}$, $R(p) > 1,\ R(s) > 1.$

Definition 1.7 The Whittaker Transform ([9] , [12] , [13])

The integral formula involving the Whittaker function is used to find the Whittaker transform is defined as

$$\int_0^\infty e^{-\frac{t}{2}} t^{\zeta-1} W_{\alpha,\beta}(t)\, dt = \frac{\Gamma\left(\frac{1}{2}+\beta+\zeta\right)\Gamma\left(\frac{1}{2}-\beta+\zeta\right)}{\Gamma(1-\alpha+\zeta)} \tag{8}$$

Where $R(\beta \pm \zeta) > -{}^{1}/_{2}$ and $W_{\alpha,\beta}(t)$ is the Whittaker confluent Hypergeometric function

$$W_{\beta,\zeta}(z) =, M_{\alpha,\beta}(z) + \frac{\Gamma(2\beta)}{\Gamma(1/2+\alpha+\beta)} M_{\alpha,-\beta}(z) \tag{9}$$

Where $M_{\alpha,\beta}(z)$ is given by ([9] p.26 , eqn. 1.150)

$$M_{\alpha,\beta}(z) = z^{1/2+\beta} e^{-1/2z}\ {}_1F_1\left(\frac{1}{2} + \beta - \alpha; 2\beta + 1; z\right) \tag{10}$$

Definition 1.8 The Hankel Transform [14] , also see [15]

The Hankel transform of $f(x)$, denoted by $g(p;v)$ is defined as

$$g(p;v) = \int_0^\infty (px)^{\frac{1}{2}}\ J_v(px) f(x)\, dx; \quad p > 0 \tag{11}$$

The following formula can be used to solve the integral in equation (see [9] , p.56-57)

$$\int_0^\infty x^{\lambda-1} J_v(ax)\, dx = 2^{\lambda-1} a^{-\lambda} \frac{\Gamma\left(\frac{\lambda+v}{2}\right)}{\Gamma\left(1+\frac{v-\lambda}{2}\right)}$$

II. INTEGRAL TRANSFORMS OF $G_{\rho,\eta,\gamma}[a,z] \times E^{\phi}_{\lambda,\beta}[z]$

Theorem 2.1: (Euler Transform)

Let $\rho,\eta,\gamma,\phi,\lambda,\beta,m,n \in C$ then Euler transform of product of special G function and Mittag – Leffler function is given as

$$B\{G_{\rho,\eta,\gamma}[a,z] \times E^{\phi}_{\lambda,\beta}[z]; m,n\} = \frac{\Gamma(n)}{\Gamma(\phi)\Gamma(\gamma)} {}_3\Psi_4\left[\left.\begin{matrix}(\gamma,1);(\phi,1);(m-1+\gamma\rho-\eta,\rho+1)\\(\rho\gamma-\eta,\rho);(\beta,\lambda);(m+\gamma\rho-\eta+n-1,\rho+1);(1,1)\end{matrix}\right| a\right] \tag{12}$$

Proof : Let I be the left hand side of (12)

$$I = B\{G_{\rho,\eta,\gamma}[a,z] \times E^{\phi}_{\lambda,\beta}[z]; m,n\}$$

By using the definition of Euler transform (6)

$$I = \int_0^1 z^{m-1}(1-z)^{n-1} G_{\rho,\eta,\gamma}[a,z] \times E^{\phi}_{\lambda,\beta}[z]\, dz$$

By using equation (3)

$$I = \int_0^1 z^{m-1}(1-z)^{n-1} z^{\gamma\rho-\eta-1} \sum_{k=0}^{\infty} \frac{(\gamma)_k (az^\rho)^k}{\Gamma(k\rho+\rho\gamma-\eta)k!} \frac{(\phi)_k}{\Gamma(\lambda k+\beta)} \frac{z^k}{k!} dz$$

$$I=\sum_{k=0}^{\infty}\frac{(\gamma)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!}\frac{(\phi)_k}{\Gamma(\lambda k+\beta)}\frac{1}{k!}\int_0^1 z^{m-1+\gamma\rho-\eta-1+k\rho+k}(1-z)^{n-1}dz$$

By using definition of beta function

$$I=\sum_{k=0}^{\infty}\frac{(\gamma)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!}\frac{(\phi)_k}{\Gamma(\lambda k+\beta)}\frac{1}{\Gamma(k+1)}B(m-1+\gamma\rho-\eta+k\rho+k,n)$$

$$I=\sum_{k=0}^{\infty}\frac{(\gamma)_k a^k(\phi)_k}{\Gamma(k\rho+\rho\gamma-\eta)k!}\frac{\Gamma(m-1+\gamma\rho-\eta+k\rho+k)\Gamma(n)}{\Gamma(\lambda k+\beta)\Gamma(k+1)\Gamma(m-1+\gamma\rho-\eta+k\rho+k+n)}$$

By using definition of Pochammer Symbol

$$I=\sum_{k=0}^{\infty}\frac{\Gamma(\phi+k)\Gamma(\gamma+k)}{\Gamma(\phi)\Gamma(\gamma)\Gamma(k\rho+\rho\gamma-\eta)}\frac{\Gamma(m-1+\gamma\rho-\eta+k\rho+k)\Gamma(n)}{\Gamma(\lambda k+\beta)\Gamma(k+1)\Gamma(m-1+\gamma\rho-\eta+k\rho+k+n)}\frac{a^k}{k!}$$

after using equation (4) we get the right hand side of (12).

Corollary 1

Use the relation between ψ function and Fox - H function (5), we get

$$B\{G_{\rho,\eta,\gamma}[a,z]\times E_{\lambda,\beta}^{\phi}[z];m,n\}=\frac{\Gamma(n)}{\Gamma(\phi)\Gamma(\gamma)}H_{3,4}^{1,3}\left[-a\,\middle|\,\begin{matrix}(1-\gamma,1);(1-\phi,1);(-m-\gamma\rho+\eta,\rho+1)\\(1-\rho\gamma+\eta,\rho);(1-\beta,\lambda);(-m-\gamma\rho+\eta-n,\rho+1)\end{matrix}\right]$$

Corollary 2

On taking $\phi=\beta=0$, the generalized Mittag Leffler function reduces to classical Mittag leffler function.

$$G_{\rho,\eta,\gamma}[a,z]\times E_{\lambda}[z]=z^{\gamma\rho-\eta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k(az^{\rho})^k}{\Gamma(k\rho+\rho\gamma-\eta)k!}\frac{1}{\Gamma(\lambda k)}\frac{z^k}{k!}$$

$$B\{G_{\rho,\eta,\gamma}[a,z]\times E_{\lambda}[z];m,n\}=\int_0^1 z^{m-1}(1-z)^{n-1}G_{\rho,\eta,\gamma}[a,z]\times E_{\lambda}[z]dz$$

$$=\int_0^1 z^{m-1}(1-z)^{n-1}z^{\gamma\rho-\eta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k(az^{\rho})^k}{\Gamma(k\rho+\rho\gamma-\eta)k!}\frac{1}{\Gamma(\lambda k)}\frac{z^k}{k!}dz$$

$$=\sum_{k=0}^{\infty}\frac{(\gamma)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!}\frac{1}{\Gamma(\lambda k)}\frac{1}{k!}\int_0^1 z^{m-1+\gamma\rho-\eta-1+k\rho+k}(1-z)^{n-1}dz$$

$$=\sum_{k=0}^{\infty}\frac{(\gamma)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!}\frac{1}{\Gamma(\lambda k)}\frac{1}{\Gamma(k+1)}B(m-1+\gamma\rho-\eta+k\rho+k,n)$$

$$=\sum_{k=0}^{\infty}\frac{(\gamma)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!}\frac{\Gamma(m-1+\gamma\rho-\eta+k\rho+k)\Gamma(n)}{\Gamma(\lambda k)\Gamma(k+1)\Gamma(m-1+\gamma\rho-\eta+k\rho+k+n)}$$

$$B\{G_{\rho,\eta,\gamma}[a,z]\times E^{\phi}_{\lambda,\beta}[z];m,n\}=\frac{\Gamma(n)}{\Gamma(\gamma)}{}_2\psi_4\left[\begin{matrix}(\gamma,1);(m-1+\gamma\rho-\eta,\rho+1)\\(\rho\gamma-\eta,\rho);(0,\lambda);(m+\gamma\rho-\eta+n-1,\rho+1);(1,1)\end{matrix}\middle|\, a\right]$$

Theorem 2.2: (Laplace Transform)

Let $\rho,\eta,\gamma,\phi,\lambda,\beta\in C$ and $R(s)>0$ then Laplace transform of product of special G function and Mittag Leffler function is given as

$$L\{G_{\rho,\eta,\gamma}[a,t]\times E^{\phi}_{\lambda,\beta}[t];s\}=\frac{1}{\Gamma(\phi)\Gamma(\gamma)s^{\gamma\rho-\eta}}{}_3\psi_3\left[\begin{matrix}(\gamma,1);(\phi,1);(\gamma\rho-\eta,\rho+1)\\(\rho\gamma-\eta,\rho);(\beta,\lambda);(1,1)\end{matrix}\middle|\,\frac{a}{s^{\rho+1}}\right] \tag{13}$$

Proof : Let I be the left hand side of (13)

$$I=L\{G_{\rho,\eta,\gamma}[a,t]\times E^{\phi}_{\lambda,\beta}[t];s\}$$

then by definition of Laplace transform (7)

$$I=\int_0^\infty e^{-st}G_{\rho,\eta,\gamma}[a,t]\times E^{\phi}_{\lambda,\beta}[t]dt$$

By using equation (3)

$$I=\int_0^\infty e^{-st}t^{\gamma\rho-\eta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k(at^\rho)^k}{\Gamma(k\rho+\rho\gamma-\eta)k!}\frac{(\phi)_k}{\Gamma(\lambda k+\beta)}\frac{t^k}{k!}dt$$

$$I=\sum_{k=0}^{\infty}\frac{(\gamma)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!}\frac{(\phi)_k}{\Gamma(\lambda k+\beta)}\frac{1}{k!}\int_0^\infty e^{-st}t^{\gamma\rho-\eta-1+k\rho+k}dt$$

$$I=\sum_{k=0}^{\infty}\frac{(\gamma)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!}\frac{(\phi)_k}{\Gamma(\lambda k+\beta)}\frac{1}{\Gamma(k+1)}\frac{\Gamma(\gamma\rho-\eta-1+k\rho+k+1)}{s^{\gamma\rho-\eta-1+k\rho+k+1}}$$

$$I=\sum_{k=0}^{\infty}\frac{\Gamma(\phi+k)\Gamma(\gamma+k)}{\Gamma(\phi)\Gamma(\gamma)\Gamma(k\rho+\rho\gamma-\eta)}\frac{\Gamma(\gamma\rho-\eta+k\rho+k)}{\Gamma(\lambda k+\beta)\Gamma(k+1)s^{\gamma\rho-\eta+k\rho+k}}\frac{a^k}{k!}$$

$$I=\frac{1}{\Gamma(\phi)\Gamma(\gamma)s^{\gamma\rho-\eta}}\sum_{k=0}^{\infty}\frac{\Gamma(\phi+k)\Gamma(\gamma+k)}{\Gamma(k\rho+\rho\gamma-\eta)}\frac{\Gamma(\gamma\rho-\eta+k\rho+k)}{\Gamma(\lambda k+\beta)\Gamma(k+1)}\left(\frac{a}{s^{\rho+1}}\right)^k\frac{1}{k!}$$

after using equation (4) we get the right hand side of (13).

Corollary 1.

On taking $\phi=\beta=0$, the generalized Mittag -Leffler function reduces to classical Mittag- leffler function

$$L\{G_{\rho,\eta,\gamma}[a,t]\times E_{\lambda}[t]\}=\int_0^\infty e^{-st}G_{\rho,\eta,\gamma}[a,t]\times E_{\lambda}[t]dt$$

$$=\int_0^\infty e^{-st} t^{\gamma\rho-\eta-1} \sum_{k=0}^\infty \frac{(\gamma)_k (at^\rho)^k}{\Gamma(k\rho+\rho\gamma-\eta)k!} \frac{1}{\Gamma(\lambda k)} \frac{t^k}{k!} dt$$

$$=\sum_{k=0}^\infty \frac{(\gamma)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!} \frac{1}{\Gamma(\lambda k)} \frac{1}{k!} \int_0^\infty e^{-st} t^{\gamma\rho-\eta-1+k\rho+k} dt$$

$$=\sum_{k=0}^\infty \frac{(\gamma)_k a^k}{\Gamma(k\rho+\rho\gamma-\eta)k!} \frac{1}{\Gamma(\lambda k)} \frac{1}{\Gamma(k+1)} \frac{\Gamma(\gamma\rho-\eta-1+k\rho+k+1)}{s^{\gamma\rho-\eta-1+k\rho+k+1}}$$

$$=\sum_{k=0}^\infty \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)\Gamma(k\rho+\rho\gamma-\eta)} \frac{\Gamma(\gamma\rho-\eta+k\rho+k)}{\Gamma(\lambda k)\Gamma(k+1)s^{\gamma\rho-\eta+k\rho+k}} \frac{a^k}{k!}$$

$$=\frac{1}{\Gamma(\gamma)s^{\gamma\rho-\eta}} \sum_{k=0}^\infty \frac{\Gamma(\gamma+k)}{\Gamma(k\rho+\rho\gamma-\eta)} \frac{\Gamma(\gamma\rho-\eta+k\rho+k)}{\Gamma(\lambda k)\Gamma(k+1)} \left(\frac{a}{s^{\rho+1}}\right)^k \frac{1}{k!}$$

$$L\{G_{\rho,\eta,\gamma}[a,t]\times E_\lambda[t]\} = \frac{1}{\Gamma(\gamma)s^{\gamma\rho-\eta}} {}_2\psi_3\left[\begin{matrix}(\gamma,1);(\gamma\rho-\eta,\rho+1) \\ (\rho\gamma-\eta,\rho);(0,\lambda);(1,1)\end{matrix}\middle| \frac{a}{s^{\rho+1}}\right]$$

Theorem 2.3: (Whittaker Transform)

Let $\rho,\eta,\gamma,\phi,\lambda,\beta \in C, R(\sigma)>0$ *and* $R(\nu\pm\sigma)>-1/2$, the Whittaker transform of product of special G function and Mittag - Leffler function is given as

$$\int_0^\infty t^{\sigma-1} e^{-\frac{\mu t}{2}} \mathrm{W}_{\alpha,\upsilon}(\mu t)\{G_{\rho,\eta,\gamma}[a,t]\times E_{\lambda,\beta}^{\phi}[t]\}dt = \left(\frac{1}{\mu}\right)^{\sigma-1+\gamma\rho-\eta} \frac{1}{\Gamma(\gamma)\Gamma(\phi)} \times$$
$$\times {}_4\psi_4\left[\begin{matrix}(\gamma,1);(\phi,1);(\upsilon+\sigma-1/2+\gamma\rho-\eta,\rho+1);(-\upsilon+\sigma-1/2+\gamma\rho-\eta,\rho+1) \\ (\beta,\lambda);(1,1);(\rho\gamma-\eta,\rho);(-\alpha+\sigma+\gamma\rho-\eta,\rho+1)\end{matrix}\middle| \frac{a}{\mu^{\rho+1}}\right] \quad (14)$$

Proof : Let I be the left hand side of (14)

$$I = \int_0^\infty t^{\sigma-1} e^{-\frac{\mu t}{2}} \mathrm{W}_{\alpha,\upsilon}(\mu t)\{G_{\rho,\eta,\gamma}[a,t]\times E_{\lambda,\beta}^{\phi}[t]\}dt$$

By using equation (3)

$$I = \int_0^\infty t^{\sigma-1} e^{-\frac{\mu t}{2}} \mathrm{W}_{\alpha,\upsilon}(\mu t)\left\{t^{\gamma\rho-\eta-1} \sum_{k=0}^\infty \frac{(\gamma)_k (at^\rho)^k}{\Gamma(k\rho+\rho\gamma-\eta)k!} \frac{(\phi)_k}{\Gamma(\lambda k+\beta)} \frac{t^k}{k!}\right\}dt$$

$$I = \sum_{k=0}^\infty \frac{(\gamma)_k (a)^k}{\Gamma(k\rho+\rho\gamma-\eta)k!} \frac{(\phi)_k}{\Gamma(\lambda k+\beta)} \frac{1}{k!} \int_0^\infty t^{\rho k+\sigma+k-1+\gamma\rho-\eta-1} e^{-\frac{\mu t}{2}} \mathrm{W}_{\alpha,\upsilon}(\mu t)dt$$

Put $\mu t=\delta$ and $dt=\dfrac{d\delta}{\mu}$, we get

$$I=\sum_{k=0}^{\infty}\frac{(\gamma)_k(a)^k}{\Gamma(k\rho+\rho\gamma-\eta)k!}\frac{(\phi)_k}{\Gamma(\lambda k+\beta)}\frac{1}{k!}\int_0^{\infty}\left(\frac{\delta}{\mu}\right)^{\rho k+\sigma+k-1+\gamma\rho-\eta-1}e^{-\frac{\delta}{2}}\mathrm{W}_{\alpha,\upsilon}(\delta)\frac{d\delta}{\mu}$$

$$I=\sum_{k=0}^{\infty}\frac{(\gamma)_k(a)^k}{\Gamma(k\rho+\rho\gamma-\eta)k!}\frac{(\phi)_k}{\Gamma(\lambda k+\beta)}\frac{1}{k!}\left(\frac{1}{\mu}\right)^{\rho k+\sigma+k-1+\gamma\rho-\eta}\int_0^{\infty}(\delta)^{\rho k+\sigma+k-1+\gamma\rho-\eta-1}e^{-\frac{\delta}{2}}\mathrm{W}_{\alpha,\upsilon}(\delta)d\delta$$

By definition of Whittaker transform (8)

$$I=\sum_{k=0}^{\infty}\frac{(\gamma)_k(a)^k}{\Gamma(k\rho+\rho\gamma-\eta)\Gamma(k+1)}\frac{(\phi)_k}{\Gamma(\lambda k+\beta)}\frac{1}{k!}\left(\frac{1}{\mu}\right)^{\rho k+\sigma+k-1+\gamma\rho-\eta}\times$$
$$\times\frac{\Gamma\left(1/2+\upsilon+\rho k+\sigma+k-1+\gamma\rho-\eta\right)\Gamma\left(1/2-\upsilon+\rho k+\sigma+k-1+\gamma\rho-\eta\right)}{\Gamma(1-\alpha+\rho k+\sigma+k-1+\gamma\rho-\eta)}$$

$$I=\sum_{k=0}^{\infty}\frac{\Gamma(k+\gamma)\Gamma(k+\phi)(a)^k}{\Gamma(\gamma)\Gamma(\phi)\Gamma(k\rho+\rho\gamma-\eta)\Gamma(k+1)\Gamma(\lambda k+\beta)}\frac{1}{k!}\left(\frac{1}{\mu}\right)^{\rho k+\sigma+k-1+\gamma\rho-\eta}\times$$
$$\times\frac{\Gamma\left(\upsilon+\rho k+\sigma+k-1/2+\gamma\rho-\eta\right)\Gamma\left(-\upsilon+\rho k+\sigma+k-1/2+\gamma\rho-\eta\right)}{\Gamma(-\alpha+\rho k+\sigma+k+\gamma\rho-\eta)}$$

after using equation (4) we get the right hand side of (14).

Corollary 1.

If we are taking $\phi=\beta=0$, then it becomes

$$\int_0^{\infty}t^{\sigma-1}e^{-\frac{\mu t}{2}}\mathrm{W}_{\alpha,\upsilon}(\mu t)\{G_{\rho,\eta,\gamma}[a,t]\times E_{\lambda}[t]\}dt$$

$$=\int_0^{\infty}{}^{\sigma-1}e^{-\frac{\mu t}{2}}\mathrm{W}_{\alpha,\upsilon}(\mu t)\sum_{k=0}^{\infty}\frac{(\gamma)_k(at^{\rho})^k}{\Gamma(k\rho+\rho\gamma-\eta)k!}\frac{1}{\Gamma(\lambda k)}\frac{t^k}{k!}dt$$

$$=\left(\frac{1}{\mu}\right)^{\sigma-1+\gamma\rho-\eta}\frac{1}{\Gamma(\gamma)}{}_3\psi_4\left[\left.\begin{matrix}(\gamma,1);\left(\upsilon+\sigma-1/2+\gamma\rho-\eta,\rho+1\right);\left(-\upsilon+\sigma-1/2+\gamma\rho-\eta,\rho+1\right)\\(0,\lambda);(1,1);(\rho\gamma-\eta,\rho);(-\alpha+\sigma+\gamma\rho-\eta,\rho+1)\end{matrix}\right|\frac{a}{\mu^{\rho+1}}\right]$$

Theorem 2.4: (Hankel Transform)

Let $\rho,\eta,\gamma,\phi,\lambda,\beta,m,n\in C$ then Hankel transform of product of special G function and Mittag – Leffler function is given as

$$g\,(p;\nu) = \frac{2^{\tau+\gamma\rho-\eta-\frac{3}{2}}\, q^{-\tau-\gamma\rho+\eta+1}}{\Gamma(\gamma)\Gamma(\phi)} \times {}_3\psi_4\left[\left.\begin{matrix} (\gamma,1);(\phi,1);\left(\dfrac{\tau+\gamma\rho-\eta-\frac{1}{2}+\nu}{2};\dfrac{\rho+1}{2}\right) \\ (\rho\gamma-\eta,\rho);(\beta,\lambda);(1,1);\left(\dfrac{\frac{5}{2}+\nu-\tau-\gamma\rho+\eta}{2};\left(\dfrac{-(\rho+1)}{2}\right)\right)\end{matrix}\right| \frac{2^{(\rho+1)}a}{q^{(\rho+1)}}\right] \tag{15}$$

Proof : Let I be the left hand side of (15)

$$I = g(p;\nu)=\int_0^\infty x^{z-1}(qx)^{\frac{1}{2}} J_\upsilon(qx)\left\{G_{\rho,\eta,\gamma}[a,x]\times E^{\phi}_{\lambda,\beta}[x]\right\}dx$$

By using equation (3)

$$I = \int_0^\infty x^{z-1}(qx)^{\frac{1}{2}} J_\upsilon(qx)\left\{x^{\gamma\rho-\eta-1}\sum_{k=0}^{\infty}\frac{(\gamma)_k (ax^\rho)^k}{\Gamma(k\rho+\rho\gamma-\eta)k!}\frac{(\phi)_k}{\Gamma(\lambda k+\beta)}\frac{x^k}{k!}\right\}dx$$

$$I = \sum_{k=0}^{\infty}\frac{(\gamma)_k (a)^k (q)^{\frac{1}{2}}(\phi)_k}{\Gamma(k\rho+\rho\gamma-\eta)k!\Gamma(\lambda k+\beta)}\frac{1}{k!}\int_0^\infty x^{k\rho+k+z+\gamma\rho-\eta-\frac{1}{2}-1} J_\upsilon(qx)dx$$

Using the following formula to solve the integral

$$\int_0^\infty x^{\lambda-1} J_\nu(ax)\,dx = 2^{\lambda-1}a^{-\lambda}\frac{\Gamma\left(\frac{\lambda+\nu}{2}\right)}{\Gamma\left(1+\frac{\nu-\lambda}{2}\right)}$$

We have

$$I = \sum_{k=0}^{\infty}\frac{(\gamma)_k (a)^k (q)^{\frac{1}{2}}(\phi)_k}{\Gamma(k\rho+\rho\gamma-\eta)k!\Gamma(\lambda k+\beta)}\frac{1}{k!}2^{k\rho+k+z+\gamma\rho-\eta-\frac{1}{2}-1} q^{-\left(k\rho+k+z+\gamma\rho-\eta-\frac{1}{2}\right)}\frac{\Gamma\left(\dfrac{k\rho+k+z+\gamma\rho-\eta-\frac{1}{2}+\upsilon}{2}\right)}{\Gamma\left(1+\dfrac{\upsilon-k\rho-k-z-\gamma\rho+\eta+\frac{1}{2}}{2}\right)}$$

$$I = \sum_{k=0}^{\infty}\frac{\Gamma(k+\gamma)\Gamma(k+\phi)2^{z+\gamma\rho-\eta-\frac{3}{2}}q^{(-z-\gamma\rho+\eta+1)}}{\Gamma(\gamma)\Gamma(\phi)\Gamma(k\rho+\rho\gamma-\eta)\Gamma(k+1)\Gamma(\lambda k+\beta)}\frac{\Gamma\left(\dfrac{z+\gamma\rho-\eta-\frac{1}{2}+\upsilon}{2}+\dfrac{k(\rho+1)}{2}\right)}{\Gamma\left(\dfrac{\upsilon-z-\gamma\rho+\eta+\frac{5}{2}}{2}-\dfrac{k(\rho+1)}{2}\right)}\left(\frac{2^{\rho+1}a}{q^{(\rho+1)}}\right)^k\frac{1}{k!}$$

after using equation (4) we get the right hand side of (15).

III. CONCLUSION

The novel conclusions gained in this study can be further adjusted in a variety of new and known integral transformations that find use in applied mathematics, bio-engineering, science, and engineering, among other fields. A few expansions of the primary findings are also taken into account.The current study yields several integral transformations that may be computed in terms of the Fox-Wright function by using the product of the Mittag Leffler function and the special G function.

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